TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 183, September 1973

PARAMETRIZATIONS OF ANALYTIC VARIETIES

BY

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ABSTRACT. Let V be an analytic subvariety of an open subset Ω of $\mathbf{C}^n$ of pure dimension r; for any $p \in V$, there exists an $n-r$ dim plane T such that $\pi_T: V \to \mathbf{C}^r$, the projection along T to $\mathbf{C}^r$, is a branched covering of finite sheeting order $\mu(V, p, T)$ in some neighborhood of V about p. π_T is called a global parametrization of V if π_T has all discrete fibers, e.g. $\dim_p V \cap (T+p) = 0$ for all $p \in V$.

Theorem. *$B = \{(p, T) \in V \times G(n-r, n) \mid \dim_p V \cap (T+p) > 0\}$ is an analytic set. If $\pi_2: V \times G \to G$ is the natural projection, then $\pi_2(B)$ is a negligible set in G.*

Theorem. *$\{(p, T) \in V \times G \mid \mu(V, p, T) \geq k\}$ is an analytic set. For each $p \in V$, there is a least $\mu(V, p)$ and greatest $m(V, p)$ sheeting multiplicity over all $T \in G$.*

If Ω is Stein, V is the locus of finitely many holomorphic functions but its ideal in $\mathcal{O}(\Omega)$ is not necessarily finitely generated.

Theorem. *If $\mu(V, p)$ is bounded on V, then its ideal is finitely generated.*

The purpose of this paper is to extend the theory of local and global parametrizations of analytic varieties and give applications to the global theory of several complex variables.

In the local theory of analytic varieties, great use is made of the local parametrization theorem which says that locally a subvariety V of $\dim r$ can be mapped into $\mathbf{C}^r$ by a finite-to-one linear function. Questions about varieties can thus be reduced to related questions about holomorphic functions on $\mathbf{C}^r$. A projection $\mathbf{C}^n \to \mathbf{C}^r$ (with respect to some basis of $\mathbf{C}^n$) is called a local parametrization of V at p, if there exists a neighborhood N of p such that $\pi \mid V \cap N$ is proper with finite fibers. In §1, several other algebraic and geometric equivalent definitions are given.

A projection $\pi: \mathbf{C}^n \to \mathbf{C}^r$ is called a global parametrization if $\pi \mid V$ has discrete fibers. Any projection $\pi: \mathbf{C}^n \to \mathbf{C}^{n-k}$ can be represented by an element $T \in G(k, n)$, the Grassmann manifold of k-dim planes in $\mathbf{C}^n$, by projection along T to a complementary subspace. The fibers of $\pi \mid V$ and $\pi^{-1}(p) = V \cap (T+p)$.

Theorem. *$B_j = \{(p, T) \in V \times G(k, n) \mid \dim_p(V \cap (T+p)) \geq j\}$ is an analytic*

Received by the editors November 6, 1972.
AMS (MOS) subject classifications (1970). Primary 32B10, 32C25.
Key words and phrases. Bounded multiplicity, finitely generated ideal.
(1) Supported by National Science Foundation Grant GU 3171.

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subvariety of $V \times G(k, n)$. *If* $\pi_2: V \times G(k, n) \to G(k, n)$ *is the natural projection, then* $\pi_2(B_j)$ *is the countable union of local varieties of codimension at least* $j(n - r - k + j)$. *It is not always possible to write this as the union of finitely many local varieties, but it is possible if* V *is an algebraic variety.*

A global parametrization is determined by any $T \in G(n - r, n)$ such that $\dim_p(V \cap (T + p)) = 0$ for all $p \in V$, in which case we say T is good for V. Considering the case $j = 1$ and $k = n - r$, we see that the set of good T forms a second category set in $G(n - r, n)$.

One might attempt to develop some more of the global theory of several complex variables as in Bishop's papers ([2], [3], [4]) if it were possible to find a linear map $\pi: \mathbf{C}^n \to \mathbf{C}^r$ expressing V as a countable increasing union of analytic covers. It is possible to do this using a holomorphic map, but this is insufficient for some applications. The best I have been able to do is to show the existence of a linear map π such that $\pi | V$ has discrete fibers. This result has applications to the extension of analytic varieties [1].

A global parametrization of a variety of pure $\dim r$ is locally an analytic cover of sheeting multiplicity $\mu(V, p, T)$ near p.

Theorem. $\{(p, T) \in V \times G(n - r, n) | \mu(V, p, T) \geq k\}$ *is an analytic subvariety of* $V \times G(n - r, n)$. *For each* $p \in V$, *there is a least and greatest multiplicity,* $\mu(p, V)$ *and* $m(p, V)$ *respectively, over all good* $T \in G(n - r, n)$.

An important result in the global theory of analytic varieties is that any subvariety V of $\mathbf{C}^n$ is the common locus of all holomorphic functions vanishing on it. Even more is true—V can be written as the locus of finitely many such functions; indeed, Forster and Ramspott have shown that any subvariety of an n-dimensional Stein space is the locus on n functions [5]. However the ideal of all holomorphic functions vanishing on V is not necessarily finitely generated. In §5, I give a geometric condition which implies the ideal is finitely generated—if the minimal multiplicity $\mu(V, p)$ is bounded on V.

This paper was the author's Ph.D. thesis at Rice University. The author would like to thank his advisor, R. O. Wells, Jr., and Reese Harvey for their constant help and encouragement, and R. C. Gunning and H. Whitney for useful correspondence.

1. Local parameterization of varieties. The results of this section are well known and can be found in [6], [7], and [11]. They are collected together for later reference.

Let $\mathbf{C}^n$ denote complex n-space. For $a \in \mathbf{C}^n$, let ${}_n\mathcal{O}_a$ denote the ring of germs of functions holomorphic in a neighborhood of a.

Let I be an ideal of ${}_n\mathcal{O}_0$, $0 \neq I \neq {}_n\mathcal{O}_0$, $0 \leq p \leq n$, and $(z_1, \cdots, z_n)$ coordinates in $\mathbf{C}^n$ with respect to an ordered basis $e_1, \cdots, e_n$. Then we have a natural homomorphism of rings ${}_p\mathcal{O}_0 \to {}_n\mathcal{O}_0/I$. It is clear that it is monic if and only if ${}_p\mathcal{O}_0 \cap I = \{0\}$.

Theorem. *The following conditions are equivalent. If any one of them holds,* $(z_1, \cdots, z_p, \cdots, z_n)$ *is said to be a semiregular coordinate system for* I.

(1) *There exist Weierstrass polynomials*

$$P_r(z_1, \cdots, z_{r-1}; z_r) \in {}_{r-1}\mathcal{O}_0[z_r] \cap I$$

for each r, $p+1 \leq r \leq n$, and ${}_p\mathcal{O}_0 \cap I = \{0\}$.

(2) ${}_n\mathcal{O}_0/I$ *is a finitely generated* ${}_p\mathcal{O}_0$ *module and* ${}_p\mathcal{O}_0 \to {}_n\mathcal{O}_0/I$ *is monic.*

(3) ${}_n\mathcal{O}_0/I$ *is integral over* ${}_p\mathcal{O}_0$ *and* ${}_p\mathcal{O}_0 \to {}_n\mathcal{O}_0/I$ *is monic.*

(4) *There exist Weierstrass polynomials*

$$P_r(z_1, \cdots, z_p; z_r) \in {}_p\mathcal{O}_0[z_r] \cap I$$

for each r, $p+1 \leq r \leq n$, *and* ${}_p\mathcal{O}_0 \cap I = \{0\}$.

(5) ${}_p\mathcal{O}_0 \cap I = \{0\}$ *and the ideal of* ${}_n\mathcal{O}_0$ *generated by* I *and* $z_1, \cdots, z_p$ *defines the point germ* $\{0\}$ *at the origin.* (0 *is an isolated point of* $(O_p \times \mathbf{C}^{n-p}) \cap \underline{V}(I)$.)

(6) ${}_p\mathcal{O}_0 \cap I = \{0\}$ *and for each* r, $p+1 \leq r \leq n$, *the ideal of* ${}_r\mathcal{O}_0$ *generated by* ${}_r\mathcal{O}_0 \cap I$ *and* $z_1, \cdots, z_p$ *defines the point germ* $\{0\}$ *in* $\mathbf{C}^r$ *at the origin.* (0 *is an isolated point of* $(O_p \times \mathbf{C}^{r-p}) \cap \underline{V}(I \cap {}_r\mathcal{O}_0)$, *where* $\underline{V}$ *is the locus in* $\mathbf{C}^r$ *of the ideal* $I \cap {}_r\mathcal{O}_0$ *of* ${}_r\mathcal{O}_0$.)

(7) $H = O_p \times \mathbf{C}^{n-p} \subset \mathbf{C}^n$ *is the maximal linear subspace of* $\mathbf{C}^n$ *satisfying* $H \cap \underline{V}(I)$ *is a point germ, i.e., if* L *is a linear subspace of* $\mathbf{C}^n$ *with* $L \supset H$ *and* $L \cap \underline{V}(I)$ *a point germ, then* $H = L$.

The proof (1) $\to$ (2) is given here as it will be needed in the last section of this paper. Let q_r = degree in z_r of P_r. Given $f \in {}_n\mathcal{O}$, since P_n does not vanish identically in z_n direction, the division theorem implies $f = g_n P_n + \Sigma_{i=0}^{q_n-1} h_{n,i} z_n^i$ with $h_{n,i} \in {}_{n-1}\mathcal{O}$ and $g_n \in {}_n\mathcal{O}$. Since $P_n \in I$, we can write $f = \Sigma h_{n,i} z_n^i \bmod I$. Now apply division theorem in ${}_{n-1}\mathcal{O}$ for each $h_{n,i}$

$$h_{n,i} = g_{n-1,i} P_{n-1} + \sum_{j=0}^{q_{n-1}-1} h_{n-1,i,j} z_{n-1}^j \quad \text{with } h_{n-1,i,j} \in {}_{n-2}\mathcal{O}.$$

Continue until the pth stage, substituting in the first equation

$$f = \sum_{\alpha_r < q_r} f_{\alpha(z_1, \cdots, z_p)} z_{p+1}^{\alpha_{p+1}} \cdots z_n^{\alpha_n} \bmod I$$

with $f_\alpha \in {}_p\mathcal{O}$. Thus ${}_n\mathcal{O}/I$ is generated by the monomials $z_{p+1}^{\alpha_{p+1}} \cdots z_n^{\alpha_n}$ with $\alpha_r < q_r$.

Remark. Note that it was not necessary for the polynomials P_r to be Weierstrass.

Definition. If $\underline{V}$ is an analytic germ, $(z_1, \cdots, z_n)$ is said to be semiregular for $\underline{V}$ if it is semiregular for the ideal $I(\underline{V})$.

There is another equivalent condition in terms of this geometric definition depending upon the fact that if the projection $\pi_r: \mathbf{C}^n \to \mathbf{C}^r$ restricted to $\underline{V}$ has discrete fibers, then $\pi_r(\underline{V})$ is an analytic germ.

(8) $O_p \times \mathbf{C}^{n-p}$ is semiregular for $\underline{V}$ if and only if $O_p \times \mathbf{C}^{r-p}$ is semiregular for $\pi_r(\underline{V})$ for all r, $p + 1 \le r \le n$.

This follows immediately from condition (6) since $\underline{V}(I \cap {}_r\mathcal{O}_0) = \pi_r(\underline{V})$. Clearly $I \cap {}_r\mathcal{O}_0 = I(\pi_r(\underline{V}))$, the ideal in ${}_r\mathcal{O}_0$ of functions vanishing on $\pi_r(V)$. Hence $\underline{V}(I \cap {}_r\mathcal{O}_0) = \underline{V}(I(\pi_r(\underline{V}))) = \pi_r(V)$, since $\pi_r(\underline{V})$ is analytic.

If $(z_1, \cdots, z_p, \cdots, z_n)$ is semiregular for $\underline{V}$, the projection $\pi: V \to \mathbf{C}^p$ is called a local parameterization of the variety. It is easy to show from definition (1) that such coordinate systems exist and $\dim \underline{V} = p$. So far we have only defined semiregular coordinates for the germ of a variety at the origin—the extension of this definition is that $(z_1, \cdots, z_n)$ are semiregular for an analytic germ $\underline{V}_a$ if they are semiregular for $\underline{V}_a + (-a) = \{z - a \mid z \in \underline{V}_a\}$. The question then arises that, if V is a subvariety of $\dim p$ of an open subset of $\mathbf{C}^n$, do there exist coordinates which are semiregular at every point of V. The answer is yes as we will see below.

Given a basis $e_1, \cdots, e_n$ of $\mathbf{C}^n$, the definition of ${}_r\mathcal{O}_0$ depends only upon the linear span of $e_{r+1}, \cdots, e_n$. Thus it is clear from conditions (2), (3), and (7) that the statement that $(z_1, \cdots, z_p, \cdots, z_n)$ is semiregular depends only upon the span of $e_{p+1}, \cdots, e_n$. Let $G(k, n)$ be the Grassmann manifold of k complex dimensional linear subspaces of $\mathbf{C}^n$. Then $G(k, n)$ is a compact complex manifold of dimension $k(n - k)$. Let $\underline{V}$ be an analytic germ of $\dim p$; an element $T \in G(n - p, n)$ is said to be good for $\underline{V}$ if $T \cap \underline{V} = \{0\}$ and T is said to be bad if it is not good.

2. Global coordinates for varieties.

2.1. Let V be a subvariety of dimension r of an open subset Ω of $\mathbf{C}^n$, $B_j = \{(p, T) \in V \times G(k, n) \mid \dim_p(V \cap (T + p)) \ge j\}$, and $\pi_2: V \times G \to G$ and $\pi_1: V \times G \to V$ the natural projections. In [8, Sätze 9–11], Grauert proved that for $j = 1$ and $k = n - r$, $\pi_2(B_1) = \{T \in G \mid T$ bad at some $p \in V\}$ is a first category set in $G(n - r, n)$. In this section we improve this result by showing that $\pi_2(B_j)$ is the countable union of local varieties of codimension at least $j(n - k - r + j)$ in $G(k, n)$. This is the best possible estimate on the codimension since if $V = \mathbf{C}^r$, any element $T \in \pi_2(B_j - B_{j+1} \cup \cdots \cup B_k)$ can be written uniquely as the direct sum of a j-dim plane T'' in $\mathbf{C}^r$ and $k - j$ dim plane T' in a fixed $n - j$ dim complementary space to T''. Hence $\dim \pi_2(B_j) \geq j(r-j) + (k-j)(n-k)$ so $\operatorname{codim} \pi_2(B_j)$ in $G(k, n) \leq j(n - r - k + j)$.

Proposition 2.1. *Let V be a subvariety of an open subset of $\mathbf{C}^n$ and $B_j = \{(p, T) \in G(k, n) \mid \dim_p(V \cap (T + p)) \geq j\}$. Then B_j is an analytic set in $V \times G(k, n)$.*

Proof. For each subset of $n - k$ elements, $H \subset \{1, 2, \cdots, n\}$, define a map $\mathbf{C}^{n-k} \to \mathbf{C}^n$ by placing 0 in each coordinate not in H. The images of these maps will be called the canonical copies of $\mathbf{C}^{n-k}$ in $\mathbf{C}^n$. For each canonical $\mathbf{C}^{n-k}$ in $\mathbf{C}^n$, consider the set $U = \{T \in G(k, n) \mid T \cap \mathbf{C}^{n-k} = \{0\}\}$. These form a covering of $G(k, n)$ and we show that $B \cap (V \times U)$ is analytic in $V \times U$. (For convenience, assume $\mathbf{C}^{n-k}$ is the last $n - k$ coordinates.)

Define a holomorphic map $f: \mathbf{C}^n \times U \to \mathbf{C}^{n-k}$ so that, for each fixed T, the map is projection along T to $\mathbf{C}^{n-k}$. Let $T \in U$ be spanned by the k row vectors $(a_{j1}, a_{j2}, \cdots, a_{jn})$ for $1 \leq j \leq k$, and A be the $n \times n$ matrix

$$\begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ a_{k1} & a_{k2} & a_{kn} \\ & & \\ & 0_{n-k \times n} & I_{n-k} \end{bmatrix}.$$

If $z_1, \cdots, z_n$ are the coordinates of a point with respect to the basis $e_1, \cdots, e_n$ and $w_1, \cdots, w_n$ are the coordinates with respect to the basis given by the rows of A, then $(w_1, \cdots, w_n) = (z_1, \cdots, z_n)A^{-1}$. The map f is given by $f(z_1, \cdots, z_n, a_{ij}) = (w_{n-k+1}, \cdots, w_n)$. Now define $\pi: V \times U \to \mathbf{C}^{n-k} \times G$ by $\pi(p, T) = (f(p, T), T)$; the fiber $\pi^{-1}\pi(p, t) = V \cap (T + p) \times T$. Thus

$$\{(p, T) \in V \times U \mid \dim_p (V \cap (T + p)) \geq j\} = \{(p, T) \in V \times U \mid \dim_{(p\,.\,T)} \pi^{-1}\pi(p, T) \geq j\}$$

is analytic. (By a theorem of Remmert [12, Satz 17] which says that if $\pi: X \to Y$ is any holomorphic map between complex spaces, then $\{x \in X \mid \dim_x \pi^{-1}\pi(x) \geq k\}$ is an analytic set.)

Lemma 2.2. *If $\dim V = r$ and $B_k \subset V \times G(k, n)$, then $\operatorname{rank} \pi_2 \mid B_k \leq k(r - k)$.*

Proof. Let $m = \operatorname{rank} \pi_2 \mid B_k$ and suppose $m > k(r - k)$. There exists a regular point of B_k where the Jacobian rank of π_2 is m. By the implicit function theorem, we may assume π_2 is of the form $(z_1, \cdots, z_\rho) \to (z_1, \cdots, z_m, 0, \cdots, 0)$, after a local biholomorphic change of coordinates in B_k and $G(k, n)$. Hence there is a holomorphic section g of π_2 over some m-dim submanifold M of an open subset of $G(k, n)$. Define a map $G: M \times \mathbf{C}^k \to \mathbf{C}^n \times G(k, n)$ by

$$G(z_1, \cdots, z_m, \alpha_1, \cdots, \alpha_k) = (\phi + \alpha_1 \Psi_1 + \cdots + \alpha_k \Psi_k, \pi_2 g)$$

where $\phi(z_1, \cdots, z_m) = \pi_1(g)$ and $\Psi(z_1, \cdots, z_m) = (\Psi_1, \cdots, \Psi_k)$; $M \to \mathbf{C}^{kn}$ is the lifting of a k plane to a basis of the plane. Then $U = G^{-1}(\Omega \times G(k, n))$ is open in $M \times \mathbf{C}^k$ and $G(U) \subset B$ by construction. We show that the map $F = \pi_1 G = \phi + \alpha_1 \Psi_1 + \cdots + \alpha_k \Psi_k : U \to V$ has Jacobian rank $> r$, contradicting the fact that $\dim V = r$.

Now $\psi: M \to \mathbf{C}^{kn}$ has rank m and ψ_i is the ith column of the $n \times k$ matrix

$$[\psi_{ij}] = \begin{bmatrix} \Psi_{11} & \Psi_{12} & & & \Psi_{1k} \\ \Psi_{21} & \Psi_{22} & & & \Psi_{2k} \\ \Psi_{n-k,1} & \Psi_{n-k,2} & & & \Psi_{n-k,k} \\ 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \\ \vdots & & & & 0 \\ 0 & & 0 & & 1 \end{bmatrix}.$$

The $n \times (m + k)$ Jacobian matrix of F is

$$\left[\frac{\partial \phi_i}{\partial z_h} + \sum_{j=1}^{k} \alpha_j \frac{\partial \Psi_{ij}}{\partial z_h} \ \vdots \ \Psi \right].$$

It suffices to show that in the expansion of some $r + 1$ subdeterminant as a polynomial in $\alpha_1, \cdots, \alpha_k$, the coefficient of the highest degree monomial is nonzero. To do this, it suffices to assume our matrix is

$$C = \left[\sum_{j=1}^{k} \alpha_j \frac{\partial \Psi_{ij}}{\partial z_h} \ \vdots \ \Psi \right].$$

This matrix is of the form

$$\begin{bmatrix} A'_{n-k\times m} & * \\ 0_{k\times m} & I_k \end{bmatrix},$$

so that rank $A' > r - k$ if and only if the column rank of $C > r$. To see the row rank of $A' > r - k$, recall that the $k(n-k)\times m$ matrix $A = [\partial\Psi_{ij}/\partial z_b]$ for $1 \le i \le n-k$, $1 \le j \le k$, $1 \le b \le m$, whose $k(i-1)+j$th row v_{ij} is $(\partial\Psi_{ij}/\partial z_1, \cdots, \partial\Psi_{ij}/\partial z_m)$, has rank m. Note that the ith row of A' is $\Sigma_{j=1}^k \alpha_j v_{ij}$. Now elementary column operations do not change the row rank of A' and a column operation on A induces a corresponding column operation on A' (but a row operation on A does not induce a row operation on A'), so we can assume A is in column echelon form.

$$\begin{bmatrix} I_{i_1} & 0 & & 0 \\ * & 0 & & 0 \\ 0 & I_{i_2} & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & I_{i_3} & \\ & & & 0 \\ 0 & & & I_{i_q} \end{bmatrix}, \qquad i_1 + i_2 + \cdots + i_q = m.$$

Since $m > k(r-k)$, there are at least $r-k+1$ different values of i, $1 \le i \le n-k$, such that there exist j and b, $1 \le j \le k$, $k(i-1)+1 \le b \le ki$, with $\partial\Psi_{ij}/\partial z_b \ne 0$. For any such i, the set of L_i of $(\alpha_1, \cdots, \alpha_k)$ satisfying $\Sigma_{j=1}^k \alpha_j \partial\Psi_{ij}/\partial z_b = 0$ for all $k(i-1)+1 \le b \le ki$ is a proper linear subspace of $\mathbf{C}^k$. Then for $(\alpha_1, \cdots, \alpha_k) \notin \bigcup L_i$, it is easily seen that row rank $(A') \ge r-k+1$.

Theorem 2.3. *Let* $\dim V = r$ *and* $A_j = \{(p, T) \in V \times G(k, n) |\ \dim_p(V \cap (T+p)) = j\}$. *Then* $\operatorname{rank} \pi_2 | A_j \le j(r-j) + (k-j)(n-k)$.

Proof. This result has already been shown for $k = j$, so we need only to reduce the problem to this case. We will take an open covering of $V \times G(k, n)$ and show the result for the restriction of π_2 for each element of the covering. For each canonical $\mathbf{C}^{n-k}$ in $\mathbf{C}^n$, recall $U = \{T \in G(k, n) |\ T$ is transversal to $\mathbf{C}^{n-k}\}$. Let $(p_0, T_0) \in A_j$ and write T_0 as the direct sum of subspaces T_0' and

T_0'' with $\dim T_0' = k - j$ and $\dim T_0'' = j$ such that the projection of $T_0 + p_0$ to $T_0'' + p_0$ along T_0' when restricted to V has discrete fibers in a neighborhood of p_0. Now take a basis $e_1, \cdots, e_n$ of $\mathbf{C}^n$ such that $e_1, \cdots, e_{k-j}$ span T_0', $e_{k-j+1}, \cdots, e_k$ span T_0'', and $e_{k+1}, \cdots, e_n$ span $\mathbf{C}^{n-k}$; let $\mathbf{C}^{n-k+j}$ denote the span of $e_{k-j+1}, \cdots, e_n$.

Each $T \in U$ is represented as a $k \times n$ matrix $[P_{k\times k} : Q_{k\times k}]$ with P nonsingular since T is transversal to $\mathbf{C}^{n-k}$. Now applying the canonical coordinate systems on the Grassmann, we see that T can be represented by

$$\begin{bmatrix} I_{k-j} & 0 & \vdots & \\ & & \vdots & P^{-1}Q \\ 0 & I_j & \vdots & \end{bmatrix}$$

and that $\dim(T \cap \mathbf{C}^{n-k+j}) = j$. Let T' be the space spanned by the first $k - j$ rows of this matrix and T'' the space spanned by the last j rows. Note that $T'' \oplus \mathbf{C}^{n-k} = \mathbf{C}^{n-k+j}$ and $T'' = T \cap \mathbf{C}^{n-k+j}$. Let $G'(k-j, n-j)$ be the subset of $G(k-j, n)$ of elements of the form $[I_{k-j} \vdots 0_{k-j\times j} \vdots *]$. Then there are holomorphic maps

$$U \to G(j, n-k+j), \qquad U \to G'(k-j, n-j),$$

$$T \to T'', \qquad\qquad T \to T' \text{ of rank } \leq (k-j)(n-k)$$

and

$$\sigma: G(j, n-k+j) \times G'(k-j, n-j) \to G(k, n),$$

$$(T'', T') \to T'' + T'.$$

Define a holomorphic map $\pi: \mathbf{C}^n \times U \to \mathbf{C}^{n-k+j}$ so that for each fixed T, the map is projection along T' to $\mathbf{C}^{n-k+j}$. Let M be the matrix

$$\begin{bmatrix} I_k & P^{-1}Q \\ 0 & I_{n-k} \end{bmatrix}.$$

If $z_1, \cdots, z_n$ are the coordinates of a point in terms of the basis $e_1, \cdots, e_n$ and $w_1, \cdots, w_n$ are the coordinates in terms of the basis given by the rows of M, then $(w_1, \cdots, w_n) = (z_1, \cdots, z_n)M^{-1}$. Let $\pi(z_1, \cdots, z_n, T) = (w_{k-j+1}, \cdots, w_n)$. Now define a map $A_j \cap (V \times U) \to \mathbf{C}^{n-k+j} \times G(k, n)$ by $(p, T) \to (\pi(p, T), T)$. The fibers of this map are $V \cap (T' + p) \times T$. By construction, the fiber at (p_0, T_0) is of dimension zero. By semicontinuity, there exists a neighborhood which we again denote $V \times U$ of (p_0, T_0) in $A_j \cap (V \times U)$ such that the restriction to $V \times U$ has fibers of dimension zero. For $(p, T), (p, S) \in V \times U$ with $T' = S'$, $\dim_p(V \cap (S' + p)) = 0$ so $\pi_T\colon V \cap (S + p) \to S'' + p$ has discrete fibers. Thus $\dim_p \pi_T(V) \cap (S'' + p) = \dim_p \pi_T(V \cap (S + p)) = j$. Now let $A'_j = \{(p, T, S) \in V \times U \times U \mid (p, T), (p, S) \in A_j \text{ and } T' = S'\}$. Then A'_j is a local variety. Consider the following diagram

$$\begin{array}{ccc}
A'_j \subset V \times U \times U & \xrightarrow{\quad \pi_2 \quad} & G(k, n) \\
\Big\downarrow g & & \Big\uparrow \sigma \\
\mathbf{C}^{n-k+j} \times G(j, n-k+j) \times U & \xrightarrow{\ f\ } & G(j, n-k+j) \times G'(k-j, n-j)
\end{array}$$

$\pi_2(p, T, S) = T$, $g(p, T, S) = (\pi(p, T), S'', T)$, $f(q, S'', T) = (S'', T')$. Note that $\sigma f g(p, T, T) = T'' + T' = T = \pi_2(p, T, T)$ so σ is onto the image of π_2. It suffices to give an upper bound for the rank of f. The fibers of g, $g^{-1}g(p, T, S) = V \cap (T' + p) \times T \times S$ are of dimension zero so in some neighborhood of (p_0, T_0, T_0), $g(A'_j) = \tilde{A}_j$ is an analytic germ. As previously noted, $(q, S'', T) \in \tilde{A}_j \Rightarrow \dim_q \pi_T(V) \cap (S'' + q) = j$ so since $\dim \pi_T(V) = r$, for fixed T, the rank of $\tilde{A}_j \xrightarrow{S''} G(j, n-k+j)$ is at most $j(r-j)$, and the rank of $\tilde{A}_j \xrightarrow{T'} G'(k-j, n-j)$ is at most $(k-j)(n-k)$. Thus rank $(f \mid \tilde{A}_j) \leq j(r-j) + (k-j)(n-k)$.

Corollary 2.4. *If* $\dim V = r$, *then* $\pi_2(B_j)$ *is the countable union of local varieties of codimension at least* $j(n-r-k+j)$ *in* $G(k, n)$.

Proof. B_j is the disjoint union of $A_j, \cdots, A_k$ and codim $\pi_2(A_l) \geq k(n-k) - [l(r-l) + (k-l)(n-k)] = l(n-r-k+l)$ so codim $\pi_2(B_j) = \min_{j \leq l \leq k} l(n-r-k+l) = j(n-r-k+j)$.

2.2. Above we show that B_j is an analytic set in $V \times G(k, n)$; we will now explicitly give the equations whose locus in $V \times G(k, n)$ is B_j.

First consider the special case $j = k = 1$ and $r = n-1$. Let $p \in V$ and in some neighborhood of p, V is the locus of $f(z) = \Sigma_\alpha f_\alpha(p)(z-p)^\alpha$, α multi-index. Now $a = [a_1, \cdots, a_n] \in \mathbf{CP}^{n-1}$ is bad for V at p if and only if for all small $\lambda \in \mathbf{C}$

$$0 = f(p + \lambda a) = \sum_\alpha f_\alpha(p)(\lambda a)^\alpha = \sum_{k=0}^{\infty} \lambda^k \sum_{|\alpha| = k} f_\alpha(p) a^\alpha$$

if and only if $0 = \Sigma_{|\alpha| = k} f_\alpha(p) a^\alpha$ for all k. These analytic equations are homogenous in a for each p, so define a subvariety of $V \times \mathbf{CP}^{n-1}$.

More generally, $B_j = \{(p, T) \in V \times G(k, n) \mid \dim_p(V \cap (T + p)) \geq j\}$; let T be represented by the $k \times n$ matrix $[a_{ij}]$ with row vectors $a_1, \cdots, a_k$. Then

$$B_j = \bigcup_{m,H,Q} \bigcap_{l,l_i,f} \text{locus of} \sum_{\substack{\alpha_1+\cdots+\alpha_j=\alpha \\ |\alpha|=l,\ |\alpha_i|=l_i}} \binom{\alpha}{\alpha_1 \cdots \alpha_j} f_\alpha(p) a_{q_1}^{\alpha_1} \cdots a_{q_j}^{\alpha_j}$$

where $l \geq 0$, $l_i \geq 0$, $l_1 + \cdots + l_j = l$, $f \in I(\underline{V}_p)$, $0 \leq m \leq k - j$, $H = \{b_1, \cdots, b_m\}$ is a subset of m elements of $\{1, \cdots, n\}$, $Q = \{q_1, \cdots, q_j\}$ is a subset of j elements of $\{1, \cdots, n\}$, $H \cap Q = \emptyset$, α is a multi-index of length n whose $(b_1, \cdots, b_m)$th coordinates are zero, $\sum_{i=1}^n a_{bi} \partial f/\partial z_i \equiv 0$ for each $b \in H$, and the binomial coefficient $\binom{\alpha}{\alpha_1 \cdots \alpha_j} = \alpha!/\alpha_1! \cdots \alpha_j!$.

Proof. Suppose $(p, T) \in B_j$ and $V \cap (T + p)$ contains a j-dim plane near p spanned by $a_{q_1}, \cdots, a_{q_j}$. Then for all $f \in I_p(V)$ and small $\lambda_1, \cdots, \lambda_j \in \mathbf{C}$

$$0 \equiv f(p + \lambda_1 a_{q_1} + \cdots + \lambda_j a_{q_j}) = \sum_\alpha f_\alpha(p)(\lambda_1 a_{q_1} + \cdots + \lambda_j a_{q_j})^\alpha$$

$$= \sum_{\substack{l_i, l \geq 0 \\ l_1+\cdots+l_j=l}} \lambda_1^{l_1} \cdots \lambda_j^{l_j} \sum_{\substack{|\alpha_i|=l_i \\ \alpha_1+\cdots+\alpha_j=\alpha}} \binom{\alpha}{\alpha_1 \cdots \alpha_j} f_\alpha(p) a_{q_1}^{\alpha_1} \cdots a_{q_j}^{\alpha_j}.$$

On the other hand, if $V \cap (T + p)$ does not contain a j-dim plane near p, there exists a m-dim plane T' in T, $m = k\text{-}\dim_p(V \cap (T + p))$ spanned by $a_{b_1}, \cdots, a_{b_m}$ such that $V \cap (T' + p)$ is discrete. Let the projection $\pi: \mathbf{C}^n \to \mathbf{C}^{n-m}$ be defined by T'; then $\pi | V$ has discrete fibers and the analytic germ $\pi(\underline{V}_p)$ contains j-dim plane near p. Hence the above equations hold in $\mathbf{C}^{n-m}$ for $f \in I(\pi(\underline{V}_p)) = I(\underline{V}_p) \cap {}_{n-m}\mathcal{O}_p$. The condition that f be independent of the variables $a_{b_1}, \cdots, a_{b_m}$ is precisely the equations $\sum_{i=1}^n a_{bi} \partial f/\partial z_i \equiv 0$.

3. Multiplicities of varieties.

3.1. It is a basic result in the local theory of analytic varieties [7] that if V is an analytic subvariety of an open subset of $\mathbf{C}^n$ of dim r and T is good for V at p, then there is a neighborhood N of p in $\mathbf{C}^n$ and N' open in $\mathbf{C}^r$ such that π_T: $N \cap V \to N'$ is an analytic cover of order k, for some integer $k > 0$. That is, π_T is a proper continuous map with finite fibers and there is a hypersurface $W \subset N'$ such that $N \cap V - \pi_T^{-1}(W)$ is dense in V and π_T: $N \cap V - \pi_T^{-1}(W) \to N' - W$ is a k-sheeted covering map.

Definition. Let $p \in V$ and $T \in G(n - r, n)$; if T is good for V at p, $\pi_T: V \to \mathbf{C}^r$ is locally an analytic cover near p of sheeting order $\mu(V, p, T)$, the multiplicity of V at p relative to T. If T is bad for V at p, we set $\mu(V, p, T) = \infty$.

Theorem 3.1. *$M_k = \{(p, T) \in V \times G(n - r, n) |\ \mu(V, p, T) \geq k\}$ is an analytic subset of $V \times G(n - r, n)$.*

Corollary 3.2. *The maximum multiplicity is bounded on compact sets. More precisely, if K is a compact subset of V, then* $\sup_{p \in K} \sup_{T \text{ good at } p} \mu(V, p, T) < \infty$. *In particular, for each $p \in V$, there is a finite maximum multiplicity* $m(V, p) = \sup_{T \text{ good at } p} \mu(V, p, T)$.

Proof. $B = \bigcap_{k=1}^{\infty} M_k$ by definition; $B \cap (K \times G)$ and each $M_k \cap (K \times G)$ lie in a compact subset of the complex space $V \times G$ and so have only finitely many irreducible components. Thus the result follows from the lemma below.

Lemma 3.3. *Let* $A = \bigcap_{i=1}^{\infty} A_i$, *each* $A_i \supset A_{i+1}$, *where A and each A_i are analytic sets with finitely many irreducible components. Then there exists k with $A = A_k$.*

Proof. Induct on m = max dim of irreducible components of A_k not in A; the conclusion of the lemma holds if $m = -1$. Choose a point x_α from each irreducible component of A_k not in A; then $\{x_\alpha\}$ is a finite set so there exists $k' > k$ with $\{x_\alpha\} \cap A_{k'} = \emptyset$. Then $m - 1 \geq$ max dim of irreducible components of A_k' not in A.

In order to prove the main theorem of this section we must first develop some more of the local theory. The first result uses two standard facts concerning the removing of singularities of holomorphic functions.

Rado's theorem. *If U is an open set in $\mathbf{C}^n$ and $h\colon U \to \mathbf{C}$ is a continuous function such that h is holomorphic on $\{z \in U \mid f(z) \neq 0\}$, then f is holomorphic on U.*

Riemann extension theorem. *If U is an open set in $\mathbf{C}^n$, A an analytic set in U, and $h\colon U - A \to \mathbf{C}$ a bounded holomorphic function, then h extends to a holomorphic function on U.*

Proposition 3.4 (Minimal analytic polynomial). *Let V be an analytic subvariety of pure* $\dim r$ *in an open subset Ω of $\mathbf{C}^n$, $\pi\colon \mathbf{C}^n \to \mathbf{C}^r$ a holomorphic map, $\Omega' = \pi(\Omega)$ and $\pi \mid V$ proper. Then for every $f \in \mathcal{O}(\Omega)$, there is a unique monic pseudopolynomial $P_f(z', t) \in \mathcal{C}(\Omega')[t]$ with $P_f(\pi(z), f(z)) \equiv 0$ on V such that P_f has minimal degree with respect to those properties. Furthermore $\{t \in \mathbf{C} \mid P_f(z', t) = 0\} = f(\pi^{-1}(z'))$.*

Proof. There is a hypersurface $A' = \pi(\mathrm{Sing}(V) \cup \{z \in \mathrm{Reg}(V) \mid \mathrm{rank}_z\, \pi < r\})$ in Ω' such that $A = \pi^{-1}(A')$ is nowhere dense in V and $\pi\colon V - A \to \Omega' - A'$ is a k-sheeted covering map. For $z' \in \Omega' - A'$, let $\pi^{-1}(z') = \{w_1(z'), \cdots, w_k(z')\}$ and λ = maximum over all $z' \in \Omega' - A'$ of the number of distinct values among $fw_1(z'), \cdots, fw_k(z')$. Let

$$h(z) = \begin{cases} \prod_{1 \le i \le j \le \lambda} (f(w_i(z')) - f(w_j(z'))) & \text{when the values are distinct,} \\ 0 & \text{when there are less than } \lambda \text{ distinct values.} \end{cases}$$

Since π is locally biholomorphic over $\Omega' - A'$, h is continuous and is holomorphic where it is nonzero. Thus by Rado's theorem, h is holomorphic on $\Omega' - A'$. Then by the Riemann extension theorem, h is holomorphic on Ω'.

Let $B' = A' \cup \{z' \in \Omega' \mid h(z') = 0\}$; for $z' \in \Omega' - B'$, f has λ distinct values on $\pi^{-1}(z')$ so define

$$P_f(z', t) = \prod_{i=1}^{\lambda} (t - f(w_i(z'))) = t^\lambda + a_1(z')t^{\lambda-1} + \cdots + a_\lambda(z')$$

where $a_i = \sigma_i(fw_1(z'), \cdots, fw_\lambda(z')) \in \mathcal{O}(\Omega' - B')$ and the σ_i are the elementary symmetric functions. Also a_i is bounded since π is proper, so that a_i extend to holomorphic functions on Ω'. It is clear that for $z' \in \Omega' - B'$, the roots of $P_f(z', t)$ are precisely the values of f on $\pi^{-1}(z')$. That this statement also holds for $z' \in B'$ follows directly from the fact that π is proper and B' is nowhere dense in Ω'.

To show P_f is unique, suppose $\tilde{P}(z', t) \in \mathcal{O}(\Omega')[t]$ is a monic pseudopolynomial of degree ν and $\tilde{P}(\pi(z), f(z)) \equiv 0$ on V. Factor $\tilde{P}$ into irreducible pseudopolynomials and let D' = union of the discriminant loci of the various factors of $\tilde{P}$. For $z' \in \Omega' - (B' \cup D')$, we can write by the fundamental theorem of algebra: $\tilde{P}(z', t) = \Pi_{j=1}^{\nu} (t - r_j(z'))$ where the λ distinct values $fw_1(z'), \cdots, fw_\lambda(z')$ must occur among the $r_j(z')$. Thus let

$$\tilde{P}(z', t) = Q(z', t) P_f(z', t) = \prod_{j=\lambda+1}^{\nu} (t - r_j(z')) \prod_{i=1}^{\lambda} (t - fw_i(z')).$$

Then the coefficients of Q are in $\mathcal{O}(\Omega' - (B' \cup D'))$ and as usual extend to Ω'. Hence P_f divides $\tilde{P}$ in the ring $\mathcal{O}(\Omega')[t]$ so if $\deg \tilde{P} = \deg P_f$, these polynomials are the same.

Remark. If V is a hypersurface, then the minimal analytic polynomial $P_n(z', z_n)$ generates the ideal $I(V)$ of all holomorphic functions vanishing on V. (See Gunning-Rossi [7, Theorem III-C-11b].) The proof of this statement is actually contained in the previous proposition since any $f \in I(V)$ can be written as a product gP where P is a pseudopolynomial vanishing identically on V.

Furthermore if V is not a hypersurface and V' is the locus in $\mathbf{C}^{r+1}$ of some $P_j(z', z_j)$, $j = r+1, \cdots, n$, then P_j generates the ideal of all functions in ${}_{r+1}\mathcal{O}$

which vanish on V', because P_j is also the minimal analytic polynomial of z_j on V' by construction.

Proposition 3.5 (Analytic polynomial). *Under the same hypothesis and notation as Proposition 3.4, for every $f \in \mathcal{O}(\Omega)$, there is a unique monic pseudopolynomial $P(z', t) \in \mathcal{O}(\Omega')[t]$ of degree k equal to the sheeting order of π such that $\{t \in \mathbf{C} \mid P(z', t) = 0\} = f(\pi^{-1}(z'))$.*

Proof. Let $P(z', t) = \Pi_{i=1}^{k} (t - f(w_i(z')))$ and the required properties follow as before. The polynomial of this proposition will be called the analytic polynomial and that of Proposition 3.4 will be called the minimal analytic polynomial. In the event that f separates the fibers of π, these polynomials will concur.

Definition. A complex linear function $L(z) = a_1 z_1 + \cdots + a_n z_n$ is said to be a regular direction if $L(z)$ separates the fibers $\pi^{-1}(z')$ over an open dense set of points, i.e., the values $Lw_1(z'), \cdots, Lw_k(z')$ are distinct for all z' in an open dense subset of Ω'. It is clear that there are lots of these; in fact, representing L by the element $a = [a_1, \cdots, a_n]$ of projective space, we have

Proposition 3.6. *Let $W = V \times G(n-r, n) - B$; then $Q = \{(p, T, a) \in V \times G(n-r, n) \times \mathbf{CP}^{n-1} \mid T$ good at p, $a \subset T$, but a not a regular direction$\}$ is an analytic set in $W \times \mathbf{CP}^{n-1}$.*

To prove this we first need

Lemma 3.7. *$\{(T, a) \in G(k, n) \times \mathbf{CP}^{n-1} \mid a \subset T\}$ is an analytic set in $G(k, n) \times \mathbf{CP}^{n-1}$.*

Proof. Let T be represented by $k \times n$ matrix with row vectors $r_1, \cdots, r_k$. There is a holomorphic embedding $G(k, n) \to \mathbf{CP}^{N-1}$, $N = \binom{n}{k}$, given by $(r_1, \cdots, r_k) \to r_1 \wedge r_2 \wedge \cdots \wedge r_k$, where $\mathbf{C}^N$ is identified with $\wedge^k \mathbf{C}^n$. Then $a \subset T$ if and only if $a \wedge r_1 \wedge \cdots \wedge r_k = 0$.

Proof of Proposition 3.6. Let $\pi: W \to \mathbf{C}^r \times G(n-r, n)$ be the holomorphic map defined by $\pi(p, T) = (\pi_T(p), T)$. The fibers of π, $\pi^{-1}\pi(p, T) = V \cap (T+p) \times T$ are discrete so π is locally an analytic cover. For each $(p, T) \in W$, there is a neighborhood U of (p, T) in W and a proper subvariety Z of the open set $\pi(U)$ such that $\pi \mid U$ is proper and $\pi: U - \pi^{-1}(z) \to \pi(U) - Z$ is a k-sheeted covering map. For $(p', T') \notin Z$, there are holomorphic functions $w_1(p', T'), \cdots, w_k(p', T')$ whose values lie in $\mathbf{C}^n$ and are the points of the set $V \cap (T' + p')$.

It is necessary to know that for no T do we have $\mathbf{C}^r \times T \subset Z$. To see this, we must explicitly define

$Z = \pi(\mathrm{Sing}\ U \cup \{$Regular points of U where the Jacobian rank of $\pi < r + r(n-r)\})$.

Now $\operatorname{Sing} U \subset \operatorname{Sing} V \times G(n-r, n)$ and for regular points $(p, T) \in U$, $\operatorname{rank}_{(p,T)}\pi = \operatorname{rank}_p \pi_T + r(n-r)$ so

$$(\mathbf{C}^r \times T) \cap Z \subset \pi_T(\operatorname{Sing}(V) \cup \{\text{Regular points of } V \text{ where rank } \pi_T < r\}) \times T.$$

This latter set has $\dim \leq r-1$, so $\mathbf{C}^r \times T \not\subset Z$.

Now if $a \subset T$ and a separates the fiber $\pi_T^{-1}(p')$, then a separates all fibers $\pi_T^{-1}(z')$ such that $\delta(z') \neq 0$, where δ is the discriminant of the minimal analytic polynomial for the linear function associated to a. Thus we have the following facts:

(1) If (p, T, a) is regular, then (p', T, a) is regular for all $(p', T) \in U$.

(2) If $(p, T) \notin Z$, $a \subset T$, and a separates $V \cap (T+p)$, then (p, T, a) is regular.

(3) If (p, T, a) is regular, then a separates $V \cap (T+p')$ for some $(p', T) \notin Z$.

It now follows that $(p, T, a) \in Q$ if and only if $a \subset T$, T is good at p, and $a \cdot (w_i(p', T) - w_j(p', T)) = 0$ for all $(p', T') \notin Z$ and some $i \neq j$.

Lemma 3.8. *Let $P(z', t)$ be the minimal analytic polynomial for z_j in $\mathcal{O}(\Omega')[t] \cap I(V)$; then*

(a) $(\partial^i P/\partial t^i)(x', x_j) = 0$ *for* $i = 0, \cdots, \lambda - 1 \Rightarrow$ *the sheeting order of π at (x', x_j) is $\geq \lambda$;*

(b) *if z_j is a regular direction and the sheeting order of π at (x', x_j) is $\geq \lambda$, then $(\partial^i P/\partial t^i)(x', x_j) = 0$ for $i = 0, \cdots, \lambda - 1$.*

Proof. (See Whitney [16, Theorem VII-8E].)

(a) $P(z', z_j) = (z_j - x_j)^k + b_1(z')(z_j - x_j)^{k-1} + \cdots + b_k(z')$, $\partial^i P(x', x_j)/\partial t^i = i!\, b_{k-i}(x')$ so $b_k(x') = \cdots = b_{k-\lambda+1}(x') = 0$. $P(x', z_j) = (z_j - x_j)^\lambda Q(x', z_j)$.

P has a zero of order λ at (x', x_j) so by continuity of the roots, for z' near x', P has λ roots near x_j. But since P is minimal, each root arises from a point of the fiber $\pi^{-1}(z')$.

(b) There are at least λ distinct values $r_i = z_j(w_i(z'))$ near x_j which are roots of $P(z', z_j)$ for most z' near x'.

$$P(z', z_j) = \prod_{i=1}^{k} (z_j - z_j w_i(z')),$$

$$\frac{\partial^m P}{\partial t^m}(z', z_j) = \sum_{\substack{H \subset \{1, \cdots, k\} \\ \#(H) = k-m}} \prod_{i \in H} (z_j - z_j w_i(z')).$$

For $m \leq \lambda - 1$, each term of the above sum has some r_i in the product, so as $(z', z_j) \to (x', x_j)$ each term of sum $\to 0$.

Lemma 3.9. *Let $P(z', t)$ be the analytic polynomial for z_j in $\mathcal{O}(\Omega')[t] \cap I(V)$; then $(\partial^i P/\partial t^i)(x', x_j) = 0$ for $i = 0, \cdots, \lambda - 1$ if and only if the sheeting order of π at (x', x_j) is $\geq \lambda$. (Proof is similar to last lemma.)*

Theorem 3.1 will be proved by first showing that $M_k - B = \{(p, T) \in V \times G - B \mid \mu(V, p, T) \geq k\}$ is analytic in $V \times G - B$, explicitly computing the equations for B to see that this set extends across B. The special case of a hypersurface will be considered first to simplify matters.

We want to express the condition of Lemma 3.5 invariantly of the direction z_n. If V is a hypersurface, then z_n is surely a regular direction. For any $f \in I(V)$, there exists $g \in \mathcal{O}(\Omega)$ so that $f(z) = g(z) P(z', z_n)$. Hence by Leibnitz's rule, π has sheeting order $\geq \lambda$ at z if and only if $(\partial^i f/\partial z_n^i)(z) = 0$ for $i = 0, \cdots, \lambda - 1$. Let $f(z) = f(p) + \sum_{j=1}^{\infty} \sum_{|\alpha|=j} f_\alpha(p) z^\alpha$ be the power series expansion of f about p; the jth directional derivative of f at p in the direction $a = (a_1, \cdots, a_n)$ is $\sum_{|\alpha|=j} f_\alpha(p) a^\alpha$. Thus $\mu(V, p, a) \geq k \Leftrightarrow \sum_{|\alpha|=j} f_\alpha(p)^\alpha = 0$ for $j = 0, \cdots, k-1$ and all $f \in I(V)$. But a is bad for V at p if and only if the same equations hold for all j and all $f \in I(V)$. Thus M_k is analytic in $V \times G$.

To handle the nonhypersurface case, consider the holomorphic map:

$$\{(T, a) \in G(n-r, n) \times \mathbf{CP}^{n-1} \mid a \subset T\} \to G'(n-r-1, n-1),$$
$$(T, a) \to T'$$

such that $T' \oplus a = T$, where $G'(n-r-1, n-1)$ is the subset of elements of $G(n-r, n)$ of the form $[I_{n-r-1}, 0_{(n-r-1)\times 1}, *_{(n-r-1)\times r}]$ (see Theorem 2.3). Consider the following diagram of holomorphic maps:

$$\begin{array}{c} V \times G(n-r, n) - B \\ \uparrow \rho \\ \{(p, T, a) \in V \times G(n-r, n) \times \mathbf{CP}^{n-1} - B \times \mathbf{CP}^{n-1} \mid a \subset T\} \\ \downarrow \pi' \\ \mathbf{C}^{r+1} \times G'(n-r-1, n-1) \times \mathbf{CP}^{n-1} \\ \downarrow \pi'' \\ \mathbf{C}^r \times G'(n-r-1, n-1) \times \mathbf{CP}^{n-1} \end{array}$$

where $\rho(p, T, a) = (p, T)$ is proper, $\pi'(p, T, a) = (\pi_{T'}(p), T', a)$, $\pi''(q, T', a) = (\pi_a(q), T', a)$ where $\pi_{T'}$ is projection along T' to $\mathbf{C}^{r+1}$ = span of $\mathbf{C}^r$ and a. Now

π' has discrete fibers and locally its image is an analytic set of pure codimension one and so is the locus of $f(z_1, \cdots, z_{r+1}, T'a)$ (the domain is a complex manifold). In fact, through use of the local parametrization π'', f can be chosen to be a pseudopolynomial in z_{r+1} so that for each fixed T' and a, $f(z, T', a)$ vanishes on $\pi_{T'}(V)$ but does not necessarily generate $I(\pi_{T'}(V)) = I(V) \cap {}_{r+1}\mathcal{O}$. Now π'' is locally an analytic cover with branches $w_1(z', T'a), \cdots, w_m(z', T', a)$ and by the construction used in Proposition 3.6, for all T' and a, $w_1(z'), \cdots, w_m(z')$ are just the distinct branches of $\pi_T\colon V \to \mathbf{C}^r$. However as $z' \to x'$, all the branches $w_i(z')$ might not converge to the same value. Let

$$f(z, T', a) = \prod_{i=1}^{m} (z_{r+1} - z_{r+1}(w_i(\pi''(z)))).$$

Let $M_k' = \{(q, T', a) \mid \Sigma_{|\alpha|=j} f_\alpha(q, T', a)a^\alpha = 0$ for all $a \subset T$ and $j = 0, \cdots, k-1\}$; then as in Lemmas 4.8 and 4.9, $M_k = \rho(\pi')^{-1}(M_k')$ so M_k is analytic in $V \times G - B$. Furthermore $(p, T) \in M_k - B$ if and only if $0 = \Sigma_{|\alpha|=j} f_\alpha(p)a^\alpha$ for $j = 0, \cdots, k-1$, all $a \subset T$ and a specific $f \in I(V) \cap {}_{r+1}\mathcal{O}$. Comparing with §2.2, which says that $(p, T) \in B$ if and only if these equations are satisfied for all j, all $a \subset T$, and all $f \in I(V) \cap {}_{r+1}\mathcal{O}$, we see that M_k extends across B.

3.2. To each point $p \in V$, there are associated a minimal and maximal multiplicity, denoted $\mu(V, p)$ and $m(V, p)$ respectively, where $\mu(V, p) = \min_{T \text{ good for } V \text{ at } p} \mu(V, p, T)$. We recover the theorem of Whitney [16, VII, 8A + 8E].

Corollary 3.10. $\{p \in V \mid \mu(V, p) \geq k\} = \bigcap_{T \in G(n-r,n)} \{p \in V \mid \mu(V, p, T) \geq k\}$ *is an analytic set in* V.

The minimal multiplicity of a point is invariant under biholomorphic mappings but the maximal multiplicity is not invariant; $\mu(V, p) = 1$ if and only if p is a regular point of V; for $p \in \mathrm{Reg}(V)$, $\mu(V, p, T) = 1$ if and only if π_T is a biholomorphism on the germ of V at p.

I do not know if $\{p \in V \mid m(V, p) \geq k\}$ is an analytic set unless $k = 2$, in which case it is analytic.

Lemma 3.11. $m(V, p) = 1$ *if and only if* V *is a complex linear subspace of* $\mathbf{C}^n$ *near* p.

Proof. Suppose $m(V, p) = 1$; then p is a regular point and let T_pV denote the tangent space to V at p. If $V \neq T_pV + p$ near p, let L be a complex line contained in T_pV such that $\dim_p(V \cap (L + p)) = 0$. There exists a $n - r$ dim plane T' containing L such that $\dim_p(V \cap (T' + p)) = 0$. Hence $\pi_{T'}$ kills L and is not a biholomorphism on V near p so $m(V, p) \geq \mu(V, p, T') > 1$, a contradiction.

It follows that $\{p \in V \mid m(V, p) = 1\}$ is just the union of the irreducible components of V which are linear subspaces of V and $\{p \in V \mid m(V, p) \geq 2\}$ is the union of the remaining components.

It is easy to compute these multiplicities for certain special cases which show that there is very little relationship between the minimal and maximal multiplicity.

Let $V_n = \{(z_1, z_2, z_3) \in \mathbf{C}^3 \mid z_2^{q_n} = z_1^{p_n} \text{ and } z_3 = n\}$ and $V = \bigcup_{n=1}^{\infty} V_n$, where p_n and q_n are relatively prime integers. Then

$$\mu(V, (z^{q_n}, z^{p_n}, n), (q_n z^{q_n - 1}, p_n z^{p_n - 1}, 0)) = m(V, (z^{q_n}, z^{p_n}, n))$$

$$= \begin{cases} \max(p_n, q_n), & z = 0, \\ 2, & z \neq 0 \end{cases}$$

and

$$\mu(V, (z^{q_n}, z^{p_n}, n), (0, 1, 0)) = \mu(V, (z^{q_n}, z^{p_n}, n)) = \begin{cases} \min(p_n, q_n), & z = 0, \\ 1 & z \neq 0. \end{cases}$$

Thus by picking each $p_n = 1$ and $q_n \longrightarrow \infty$, we have an example with the minimal multiplicity bounded by 1 and the maximal multiplicity unbounded on a variety.

There is also such an example with V irreducible. Let $\{a_n\}$ be any discrete sequence of points in the complex plane $\mathbf{C}$; by the Weierstrass theorem, there is an entire function f with zeros of order $p_n - 1$ at the points a_n. Let g be an entire function whose derivative is f and let A be the image of the function: $\mathbf{C} \longrightarrow \mathbf{C}^2$ given by $t \longrightarrow (t, g(t))$; then A is a complex manifold so the minimal multiplicity is indentically one on A. Near the points $(a_n, g(a_n))$, A looks like $z_1^{p_n} = z_2$ so has maximal multiplicity p_n.

Proposition 3.12. *The maximal multiplicity is bounded on an algebraic variety; more precisely if V is an algebraic subvariety of $\mathbf{C}^n$ of pure dim r, then there exists an integer $K > 0$ such that $\mu(V, p) \leq K$ for all $p \in V$.*

Proof. First consider the hypersurface case—let V be an algebraic subvariety of pure codim one. Then V can be expressed globally as an analytic cover over $\mathbf{C}^{n-1}$. Let $P(z', z_n)$ be the analytic polynomial for z_n; the coefficients of P are entire functions with polynomial growth and so are polynomials (see Rudin [13, Theorems 1 and 2]). Let $\deg P$ denote the homogeneous degree of P. It follows that $m(V, p) \leq \deg P$ for all $p \in V$ since: a bad at $p \in V \Longleftrightarrow a$ satisfies all the homogenous polynomials $\sum_{|\alpha| = j} f_\alpha(p) a^\alpha$ for all $f \in I(V)$; $\mu(V, p, a) \geq k \Longleftrightarrow a$ satisfies the first $k - 1$ homogenous polynomials for all $f \in I(V) \Longleftrightarrow a$ satisfies the first $k-1$ homogeneous polynomials of P, since P generates $I(V)$.

More generally, a careful examination of the proof of Theorem 4.1 shows that M_k is an algebraic subvariety of $\mathbf{C}^n \times G(n-r, n)$: $\pi'(V \times G(n-r, n) \times \mathbf{CP}^{n-1} - B \times \mathbf{CP}^{n-1})$ is contained in an algebraic subvariety of pure codim one. By use of the parametrization π'', $f(z_1, \cdots, z_{r+1}, T', a)$, the analytic polynomial of z_{r+1} is actually a polynomial and $M_k = \{(p, t) | \sum_{|\alpha|=j} f_a(p, T', a)a^\alpha = 0$ for all $a \subset T$ and $j = 0, \cdots, k-1\}$. Thus $m(V, p) \leq \deg f$.

Remark. It is well known that the polynomial ring $\mathbf{C}[z_1, \cdots, z_n]$ is Noetherian, so the ideal of all polynomials vanishing on V is generated by finitely many elements $P_1, \cdots, P_m$. It follows that $P_1, \cdots, P_m$ also generate the ideal of all holomorphic functions vanishing on V [13, Theorems 4.1, 4.4, and 5]. However if V is not an algebraic variety, this construction fails and it is not necessarily true that the ideal of all holomorphic functions vanishing on V is finitely generated.

3.3. Let $\mathcal{C}_p(V)$ be the Whitney tangent cone to V at p ([14], [15]); $a \in \mathcal{C}_p(V)$ if and only if there exist $q_i \in V$ with $q_i - p$ converging to a in $\mathbf{CP}^{n-1}$. Equivalently $\mathcal{C}_p(V)$ is the common locus of the initial polynomials of all holomorphic functions vanishing on V near p. $\mathcal{C}_p(V)$ is an analytic variety of the same dimension as V and $\mu(V, p, T) = \mu(V, p)$ if and only if $\dim(\mathcal{C}_p(V) \cap T) = 0$.

Let $P(z', z_j)$ be the minimal analytic polynomial for z_j; then $\mu(V, p, T) \geq$ degree of P in $z_j \geq$ order of P.

Proposition 3.13. *If z_j is a regular direction, then* $\dim(\mathcal{C}_p(V) \cap T) \leq 1 \Leftrightarrow$ $\operatorname{ord} P = \mu(V, p)$.

Proof. First consider the hypersurface case, where the first condition is automatically satisfied since $\dim T = 1$; P generates the ideal $I(V)$ and the order of products is the sum of orders, so P has minimal order with respect to elements of $I(V)$. Choosing a direction $T \in \mathbf{CP}^{n-1}$ with $\mu(V, p, T) = \mu(V, p)$, we see that $\operatorname{ord} P = \mu(V, p)$.

More generally, T can be written as $T' \oplus L$, where T is a $n-r-1$ dim plane and L is the complex line determined by z_j. Now the linear function z_j and hence $P(z', z_j)$ do not depend upon the choice of T', so pick T' so that $\dim(\mathcal{C}_p(V) \cap T') = 0$. Let $\mathbf{C}^{r+1} = \mathbf{C}^r \oplus L$, V' be the locus in $\mathbf{C}^{r+1}$ of P, D be the locus in $\mathbf{C}^r$ of the discriminant of P, and $D' = \pi_L^{-1}(D)$. Then P_j is also the minimal analytic polynomial of V', $\pi_{T'}(V) = V'$, and $\pi_{T'}: V - \pi_{T'}^{-1}(D') \to V' - D'$ is one-to-one since z_j is a regular direction. For any local parametrization $\pi_a: V' \to \mathbf{C}^r$, $a \in \mathbf{CP}^{n-1}$: $\mu(V', p, a) = (\#$ sheets of $\pi_{T'}) \cdot (\#$ sheets of $\pi_a) =$ $\mu(V, p, T' \oplus a) \geq \mu(V, p)$.

Hence by the above, we have $\operatorname{ord} P = \mu(V', p) \le \mu(V', p, a) = \mu(V, p, T' \oplus a) \ge \mu(V, p)$. If a is chosen so that $a \notin \mathcal{C}_p(V') = \pi_{T'}(\mathcal{C}_p(V))$, then $\dim(\mathcal{C}_p(V) \cap (T' \oplus a)) = 0$, so $\operatorname{ord} P = \mu(V, p)$.

4. Finitely generated varieties. As remarked earlier, the ideal of a variety is not necessarily finitely generated. In this section, I give an example of a variety that is not finitely generated and a geometric condition which implies the ideal is finitely generated.

Let V be an analytic subvariety of pure $\dim r$ of an open subset Ω of $\mathbf{C}^n$. Denote by $I(V, p)$ the ideal in ${}_n\mathcal{O}_p$ of all functions vanishing on the germ of V at p and by $I(V)$ the ideal in $\mathcal{O}(\Omega)$ of functions vanishing on V. Let $\# I(V, p)$ denote the minimal number of generators of $I(V, p)$ over ${}_n\mathcal{O}_p$, which is unique by Nakayama's lemma since ${}_n\mathcal{O}_p$ is a local Noetherian ring.

Theorem 4.1. $\# I(V, p) \le [\mu(V, p) + 1]^n$. *If* $r = 1$ *or* 2, $\# I(V, p) \le 2(n - r)[\mu(V, p) + 1]^{n-r}$.

This local result can be combined with the following theorem of Kripke [9, Theorem 1] to get global results—$I(V)$ is finitely generated if the minimal multiplicity of V is bounded.

Theorem. *Let X be an r-dim analytic space and $\mathfrak{F}$ a coherent analytic sheaf on X such that the global sections $\mathfrak{F}(X)$ generate each stalk $\mathfrak{F}_x$. If $\# \mathfrak{F}_x \le k$ for every $x \in X$, then $\{(s_1, \cdots, s_{k(r+1)}) \in \mathfrak{F}^{k(r+1)}(X) \mid s_1, \cdots, s_{k(r+1)}$ generate $\mathfrak{F}(X)\}$ is the complement of a first category set in $\mathfrak{F}^{k(r+1)}(X)$.*

Corollary 4.2. *Also assume that Ω is a domain of holomorphy. If $\mu(V, p) < K$ for all $p \in V$, then $\# I(V) \le (r + 1) K^n$. If $r = 1$ or 2, $\# I(V) \le 2(r + 1)(n - r) K^{n-r}$.*

Remark. In the above corollary, the assumption about Ω can be weakened—instead of Ω being a domain of holomorphy, it is enough to have $V = \bigcap_{f \in I(V)} \operatorname{locus}(f)$ because it then follows that there are enough global functions to generate each stalk $I(V, p)$ of the coherent sheaf $\mathfrak{I}(V)$. Let $\widetilde{\Omega}$ be the envelope of holomorphy of Ω and $\widetilde{V}$ be the common locus of all $f \in I(V) \subset \mathcal{O}(\Omega) = \mathcal{O}(\widetilde{\Omega})$. Then $\widetilde{\Omega}$ is a Stein manifold so by Cartan's Theorem A, $I(V)$ generates $I(\widetilde{V}, p)$ for all $p \in \widetilde{V}$.

The proof of Theorem 4.1 is basically a big chase through the proof of Oka's lemma—coherence of $\mathcal{O}$. The ideal $I(V, p)$ can be viewed as the sheaf of relations between some special functions and the proof of Oka's lemma as an algorithm for computing the generators. The bound on the multiplicity enables one to put a bound on the number of resulting generators.

Let $p \in V$, and $\pi: \mathbf{C}^n \to \mathbf{C}^r$ be a projection expressing V in a neighborhood of p as an analytic cover of minimal sheeting order $k = \mu(V, p)$. Since for $T \in G(n-r, n)$ near $\mathbf{C}^{n-r}$, $\mu(V, p, T) = \mu(V, p, \mathbf{C}^{n-r}) = k$, we can also assume that π has no branches which are identically zero. Also choose π so that $\mathbf{C}^r$ is not contained in the tangent cone to V at p, e.g., $\dim(\mathbf{C}^r \cap \mathcal{C}_p(V)) \leq r-1$. Now π is also a parametrization of $\mathcal{C}_p(V)$ and one can apply the usual theory of analytic covers to it as well as V.

Choose a basis of $\mathbf{C}^{n-r}$ so that $z_{r+1}, \cdots, z_n$ are all regular directions for both $\pi\colon V \to \mathbf{C}^r$ and $\pi\colon \mathcal{C}_p(V) \to \mathbf{C}^r$; this can be done since the set of nonregular directions forms a linear subvariety of $\mathbf{CP}^{n-1}$ by Proposition 3.6. It is also possible to choose this basis so that no z_j, $r+1 \leq j \leq n$, vanishes identically on any branch of π on V or $\mathcal{C}_p(V)$. Let $P_j(z', z_j) = z_j^k + a_{1j}(z')z_j^{k-1} + \cdots + a_{kj}(z')$ be the analytic polynomial for z_j of V. Then the above conditions insure that $a_{kj}(z') \not\equiv 0$, since a_{kj} is the product of the values of z_j on the branches of π.

Lemma 4.3. *There exists a basis of* $\mathbf{C}^r$ *so that the tail coefficients* $a_{kj}(z')$ *vanish to order no more than* $\mu(V, p)$ *in each direction* $z_1, \cdots, z_r$.

Proof. Let V' be the locus of P_j in $\mathbf{C}^{r+1}$ = span of $\mathbf{C}^r$ and z_j, and let T' be a $n-r-1$ dim plane in $\mathbf{C}^{n-r}$ such that $\mathbf{C}^n = \mathbf{C}^{r+1} \oplus T'$. For any direction $L = c_1z_1 + \cdots + c_{r+1}z_{r+1}$ in $\mathbf{C}^{r+1}$ which is good for V', $\mu(V', p, L) = \mu(V, p, T' \oplus L) \leq m(V, p)$. For any direction L in $\mathbf{C}^r$, $\mu(V', p, L) \geq l \Leftrightarrow P_j$ vanishes to order $\geq l$ in the direction $L \Leftrightarrow a_{kj}(z')$ vanishes to order $\geq l$ in the direction L. Now $a_{kj} \not\equiv 0$, so a_{kj} vanishes to order $\leq m(V, p)$ in any direction which is a nonsolution to the initial polynomial of a_k.

But it is also possible to choose direction L in $\mathbf{C}^r$ so that $\dim(\mathcal{C}_p(V) \cap (T' \oplus L)) = 0$; then $\mu(V', p, L) = \mu(V, p)$ and so a_{kj} vanishes to order only $\mu(V, p)$ in the direction L. Such directions are constructed as follows: Now $\dim(T' \cap \mathcal{C}_p(V)) = 0$ so $\pi_{T'}(\mathcal{C}_p(V))$ is analytic in $\mathbf{C}^{r+1}$ and of pure dim r. Also $\dim(\mathbf{C}^r \cap \pi_{T'}(\mathcal{C}_p(V))) \leq r-1$ because z_j vanishes identically on no branch of π on $\mathcal{C}_p(V)$. Since $\dim(\mathcal{C}_p(V) \cap (T' \oplus L)) = 0$ is equivalent to $\dim(L \cap \pi_{T'}(C_p(V))) = 0$, there are lots of such directions.

This basis of $\mathbf{C}^r$ can also be chosen to satisfy the additional condition that for any $a_{ij} \not\equiv 0$, a_{ij} does not vanish identically in any of the directions $z_1, \cdots, z_r$, by choosing directions which are nonsolutions to the respective initial polynomials.

Later on it will be necessary to write each $a_{ij}(z')$ as a unit times a Weierstrass polynomial in z_r with coefficients in ${}_{r-1}\mathcal{O}_p$, and then write each resulting coefficient again as a unit times a Weierstrass polynomial in z_{r-1} with coefficients in ${}_{r-2}\mathcal{O}_p$, etc. The above conditions on the basis of $\mathbf{C}^r$ guarantee that this is possible and that for a tail coefficient

$$a_{kj}(z_1, \cdots, z_r) = u(z_1, \cdots, z_r)$$
$$\cdot\, [z_r^l + a_{1kj}(z_1, \cdots, z_{r-1})z_r^{l-1} + \cdots + a_{lkj}(z_1, \cdots, z_{r-1})],$$

$l \le \mu(V, p)$ and again a_{lkj} vanishes to order $\le \mu(V, p)$ in the directions $z_1, \cdots, z_{r-1}$.

Now with the fixed basis of $\mathbf{C}^n$ determined above, we proceed to construct some canonical functions of the variety.

Lemma 4.4. *For every $f \in {}_n\mathcal{O}_p$, there is a unique pseudopolynomial $Q_f \in {}_r\mathcal{O}_p[t]$ of degree $\le k - 1$ such that $f(z)\delta(\pi(z)) - Q_f(\pi(z), z_{r+1}) \in I(V, p)$, where δ is the discriminant of the polynomial $P_{r+1}(z', z_{r+1})$. In particular, there are $Q_j \in {}_r\mathcal{O}_p[z_j]$ for $j = r + 2, \cdots, n$ so that $z_j\delta(z') - Q_j(z', z_{r+1}) \in I(V, p)$.*

This is proved by algebraic methods in [11, Lemma 2, p. 35] for irreducible varieties and by geometric means in [6, Theorem 18, p. 113] for pure dim varieties as follows:

$$\delta(z') = \prod_{i \ne j} (z_{r+1}(w_i(z')) - z_{r+1}(w_j(z'))) \not\equiv 0$$

as z_{r+1} is a regular direction, where the w_i are the branches of π. Now for any $f \in {}_n\mathcal{O}$, we want to find $b_j \in {}_r\mathcal{O}$ such that

$$\delta(z')f(w_i(z')) = \sum_{j=0}^{k-1} b_j(z')z_{r+1}(w_i(z'))^j \quad \text{for } i = 1, \cdots, k.$$

These equations can be viewed as a system of k linear equations in the k unknown values $b_j(z')$; hence by Cramer's rule

$$b_j(z') \det[1, z_{r+1}w_i(z'), \cdots, z_{r+1}w_i(z')^{k-1}]$$
$$= \det[1, z_{r+1}w_i(z'), \cdots, z_{r+1}w_i(z')^{j-1},$$
$$\delta(z')fw_i(z'), z_{r+1}w_i(z')^{j+1}, \cdots, z_{r+1}w_i(z')^{k-1}].$$

The determinant appearing in the left-hand side is the van der Monde determinant Δ, and it is well known that $\Delta^2 = \delta$. Factoring δ out of the right-hand side of the equations produces the explicit formula

$$b_j = \Delta \cdot \det[1, z_{r+1}(w_i), \cdots, z_{r+1}(w_i)^{j-1}, f(w_i), z_{r+1}(w_i)^{j+1}, \cdots, z_{r+1}(w_i)^{k-1}].$$

Lemma 4.5. *For $N \geq (k-1)(n-r-1)$, $f \in I(V, p)$ if and only if $\delta^N f$ lies in the ideal of ${}_n\mathcal{O}_p$ generated by $P_{r+1}, \cdots, P_n, z_{r+2}\delta - Q_{r+2}, \cdots, z_n\delta - Q_n$.*

This is proven for irreducible varieties in [11, Lemma 4, p. 37 or Theorem 5, p. 77]. The same proof works for varieties of pure dim:

Recall from §1 that ${}_n\mathcal{O}_p/I(V, p)$ is finitely generated over ${}_r\mathcal{O}_p$, i.e. for every $f \in {}_n\mathcal{O}$,

$$f = \sum_{\alpha_j < k} f_\alpha(z_1, \cdots, z_r) z_{r+1}^{\alpha_{r+1}} \cdots z_n^{\alpha_n} \mod (P_{r+1}, \cdots, P_n).$$

Then

$$\delta^N f = \tilde{R}(z', z_{r+1}) \mod (P_{r+1}, \cdots, P_n, z_{r+2}\delta - Q_{r+2}, \cdots, z_n\delta - Q_n)$$

where $\tilde{R} \in {}_r\mathcal{O}[z_{r+1}]$ is a pseudopolynomial, by replacing $z_i\delta$ by $(z_i\delta - Q_j) + Q_j$ for $j = r+2, \cdots, n$. Now applying the algebraic division algorithm to $\tilde{R}$ and P_{r+1}, we have

$$\delta^N f = R(z', z_{p+1}) \mod (P_{r+1}, \cdots, P_n, z_{r+2}\delta - Q_{r+2}, \cdots, z_n\delta - Q_n)$$

where $R \in {}_r\mathcal{O}[z_{r+1}]$ is of degree $< \deg P_{r+1}$. Now if $f \in I(V, p)$, then $R \in I(V, p)$ so $R \equiv 0$ because it has lower degree than the minimal polynomial which is unique by Proposition 4.4.

Conversely if $\delta^N f$ is generated by these functions, then $\delta^N f \equiv 0$ on V; but $\{z \in V \mid \delta(z') \neq 0\}$ is dense in V so $f \equiv 0$ on V.

I now sketch the proof of Oka's lemma as in [11, Theorem 4, p. 77] as it will be necessary to make some careful observations about the proof. I ignore the question of whether all the steps can be carried out in some fixed open set, which was the original motivation for the lemma.

Oka's lemma. *Let $f_1, \cdots, f_q \in {}_n\mathcal{O}$ and $\mathfrak{R}(f_1, \cdots, f_q)$ denote the sheaf of relations, i.e., $\mathfrak{R}$ is the submodule of ${}_n\mathcal{O}^q$ consisting of $(\alpha_1, \cdots, \alpha_q) \in {}_n\mathcal{O}^q$ with $\Sigma_{i=1}^q \alpha_i f_i = 0$. Then $\mathfrak{R}$ is finitely generated over ${}_n\mathcal{O}$.*

The proof is by induction on n. The relations of $f_1, \cdots, f_q$ will be reduced to several relations of the type $\mathfrak{R}(g_1, \cdots, g_q)$ in ${}_{n-1}\mathcal{O}^q$.

We may clearly suppose that at least one $f_i \neq 0$. Since the result is local and permits multiplication by units, we can assume, after a linear change of coordinates if necessary, that each f_i is a pseudopolynomial,

$$f_i = \sum_{\nu=0}^{k_i} a^i_\nu(z_1, \cdots, z_{n-1}) z_n^\nu \quad \text{where } a^i_\nu \in {}_{n-1}\mathcal{O},$$

and at least one f_i, say f_q, is a Weierstrass polynomial of degree k. A relation $(\alpha_1, \cdots, \alpha_q) \in \mathfrak{R}$ is said to be a polynomial relation if each $\alpha_i \in {}_{n-1}\mathcal{O}[z_n]$; then $\mathfrak{R}$ is generated over ${}_n\mathcal{O}$ by the polynomial relation.

Let $(\alpha_1, \cdots, \alpha_q) \in \mathfrak{R}$ and for each $i = 1, \cdots, q-1$, write $\alpha_i = \mu_i f_q + r_i$ by the division theorem, where $\mu_i \in {}_n\mathcal{O}$ and $r_i \in {}_{n-1}\mathcal{O}[z_n]$ has degree $< \deg f_q$. Let r_q be defined by the equations:

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_q \end{pmatrix} = \mu_1 \begin{pmatrix} f_q \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ -f_1 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ f_q \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ -f_2 \end{pmatrix} + \cdots + \mu_{q-1} \begin{pmatrix} 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \\ f_q \\ -f_{q-1} \end{pmatrix} + \begin{pmatrix} r_1 \\ r_2 \\ \cdot \\ \cdot \\ \cdot \\ r_{q-1} \\ r_q \end{pmatrix}.$$

It remains only to show that r_q is a pseudopolynomial. Clearly $(r_1, \cdots, r_q) \in \mathfrak{R}$ so $r_q f_q = -\sum_{i<q} r_i f_i \in {}_{n-1}\mathcal{O}[z_n]$. By the algebraic division algorithm $r_q f_q = Q f_q + R$ where $Q, R \in {}_{n-1}\mathcal{O}[z_n]$ and $\deg R < \deg f_q$. But then R/f_q is holomorphic, f_q vanishes to order $\deg f_q$ in the z_n direction because it is Weierstrass, and R vanishes to lower order. Thus $R \equiv 0$ and $r_q = Q$.

Remark 1. The first $q-1$ entries of the above relations are all of degree $\leq \deg f_q$. However, little can be said about the degree of the qth entries of these relations other than they have degree $\leq \max_{1 \leq i \leq q} \deg f_i = K$. In the application to the structure sheaf of a variety, f_q will always be picked to have degree $\leq \mu(V, p)$.

Next we show that there exist finitely many polynomial relations $\pi = (\pi_1, \cdots, \pi_q)$ with $\deg \pi_i \leq k$ for $i < q$ and $\deg \pi_q \leq K$, which generate $\mathfrak{R}$ over ${}_n\mathcal{O}$. Let $\pi_i = \sum_\nu c^i_\nu z^\nu_n$ and $f_i = \sum^{k_i}_\nu a^i_\nu z^\nu_n$; then π is a relation if and only if

$$(*) \qquad \sum_{i=1}^{q} \sum_{j=0}^{\nu} a^i_{\nu-j} c^i_j = 0 \quad \text{in } {}_{n-1}\mathcal{O} \text{ for } \nu = 0, \cdots, K+k.$$

This means the element

$$[c^i_\nu] = (c^1_0, \cdots, c^1_k, c^2_0, \cdots, c^2_k, \cdots, c^{q-1}_0, \cdots, c^{q-1}_k, c^q_0, \cdots, c^q_K) \in {}_{n-1}\mathcal{O}^m,$$

$m = (k+1)(q-1) + K + 1$, is a relation between the finitely many sections

$$s_\nu = (a^1_\nu, \cdots, a^1_{\nu-k}, a^2_\nu, \cdots, a^2_{\nu-k}, \cdots, a^{q-1}_\nu, \cdots, a^{q-1}_{\nu-k}, a^q_\nu, \cdots, a^q_{\nu-K}),$$

where $a^i_\nu = 0$ if $\nu < 0$. Thus by the induction hypothesis, there exist finitely many sections $[b^i_\nu]_\mu \in {}_{n-1}\mathcal{O}^m$ generating these relations. That is for any $[c^i_\nu]$ as above, there exist $\psi_\mu \in {}_n\mathcal{O}$ such that $c^i_\nu = \Sigma_\mu \psi_\mu b^i_{\nu\mu}$ for all i and ν. Thus

$$\pi = [\pi_i] = \left[\sum_\nu c^i_\nu z^\nu_n\right] = \sum_\mu \psi_\mu \left[\sum b^i_{\nu\mu} z^\nu_n\right]$$

Remark 2. Each s_ν has as entries either all the tail coefficients a^i_0 or $a^q_k = 1$. (For $\nu \leq k$, it contains a^i_0 and for $\nu \geq k$, it contains a^q_k.)

Remark 3. There is no mixing of terms with respect to i in equations (*), e.g., only a^i terms are multiplied times c^i terms.

Now we apply this method to the relations in ${}_n\mathcal{O}^{2(n-r)}$ between δ^N, $z_{r+2}\delta - Q_{r+2}, \cdots, z_n\delta - Q_n, P_{r+1}, \cdots, P_n$ to reduce it to some other relations of functions of less variables and repeat the process on each of these relations until we come down to $(2k)^{n-r}$ relations of $2(n-r)(k+1)^{n-r}$ functions of r variables. Indeed, during the step from a relation in ${}_n\mathcal{O}^{2(n-r)}$ to $2k$ relations in ${}_{n-1}\mathcal{O}^{2(n-r)(k+1)}$, use P_n as the Weierstrass element f_q to divide by in the reduction to polynomial relations; $P_{n+1}, \cdots, P_{n-1}, z_{r+2}\delta - Q_{r+2}, \cdots, z_{n-1}\delta - Q_{n-1}$ are all of degree zero in z_n and $z_n\delta - Q_n$ is of degree one in z_n so $k = K$. By Remark 2, each resulting section so contains either $P_{r+1}, \cdots$, and P_n or 1—in any event, some entry whose degree in z_{n-1} is $\leq \mu(V, p)$. In the next stage the choice of special element to reduce the section depends upon ν: for $\nu \leq k$, use P_{n-1}; for $\nu \geq k+1$, use 1. Again each resulting section will contain some entry whose degree in z_{n-2} is $\leq \mu(V, p)$, etc.

The special results for $r = 1$ and 2 arise because relation ideals in ${}_1\mathcal{O}$ and ${}_2\mathcal{O}$ are essentially trivial.

Proposition 4.5. *Let* $\lambda\colon {}_2\mathcal{O}^l \to {}_2\mathcal{O}^{l'}$ *be an* ${}_2\mathcal{O}$*-homomorphism, then* $\#\ker\lambda \leq l$.

Proof (as in [9, Proposition 8]). Consider the exact sequence

$$\operatorname{Ker}\lambda \to {}_2\mathcal{O}^l \xrightarrow{\lambda} {}_2\mathcal{O}^{l'} \to {}_2\mathcal{O}^{l'}/\operatorname{im}\lambda \to 0.$$

According to the Hilbert-Syzygy theorem [7, p. 74], $\operatorname{Ker}\lambda$ is a free ${}_2\mathcal{O}$ module. Since $\operatorname{Ker}\lambda$ is a free submodule of ${}_2\mathcal{O}^l$, $\#\operatorname{Ker}\lambda \leq l$.

Proposition 4.6. *Let* $\lambda\colon {}_1\mathcal{O}^l \to {}_1\mathcal{O}^{l'}$ *be a* ${}_1\mathbb{C}$ *homomorphism, then* $\#\operatorname{Ker}\lambda \leq l$.

Proof. First suppose $l' = 1$ and $\lambda = (\lambda_1(z), \cdots, \lambda_l(z))$. Each $\lambda_j(z)$ can be written as $z^{d_j}u_j(z)$ where $u_j(0) \neq 0$ and by relabeling, assume that $d_1 = \min(d_1, \cdots, d_l)$. Then $(f_1, \cdots, f_l) \in \operatorname{Ker}\lambda$ if and only if

$$z^{d_1}\left(u_1 f_1 + \sum_{j=2}^{l} z^{d_j - d_1} u_j f_j\right) = 0.$$

Consequently the mapping ${}_1\mathcal{O}^{l-1} \to {}_1\mathcal{O}^{l}$ defined by

$$(g_2, \cdots, g_l) \to \left(-\frac{1}{u_1}\sum_{j=2}^{l} z^{d_j - d_1} u_j g_j,\ g_2, \cdots, g_l\right)$$

is onto $\operatorname{Ker}\lambda$.

Now induct on l' and let λ be the $l' \times l$ matrix $[\lambda_{ij}]$. By the case $l' = 1$, the solutions to $\Sigma_{j=1}^{l}\lambda_{1j}f_j = 0$ have a basis (Φ_j^{μ}), $\mu = 1, \cdots, l$. Now the general solution $(f_j) \in \operatorname{Ker}\lambda$ has the form $f_j = \Sigma_{\mu}\psi_{\mu}\Phi_j^{\mu}$ and satisfies the 2nd through lth equations, so

$$0 = \sum_j \lambda_{ij} \sum_{\mu} \psi_{\mu}\Phi_j^{u} = \sum_{\mu} \psi_{\mu}\left(\sum_j \lambda_{ij}\Phi_j^{\mu}\right) \quad \text{for } i = 2, \cdots, l'.$$

Let $\gamma_{i\mu} = \Sigma_j\lambda_{ij}\Phi_j^{\mu}$ for $i = 2, \cdots, l'$. Then by the induction hypothesis, the solutions to $0 = \Sigma_{\mu}\gamma_{i\mu}\psi_{\mu}$ have a basis (X_{μ}^{ρ}), $\rho = 1, \cdots, l$, so $\psi_{\mu} = \Sigma_{\rho}w_{\rho}X_{\mu}^{\rho}$ and $\Phi_j = \Sigma_{\rho}w_{\rho}\Sigma_{\mu}X_{\mu}^{\rho}\Phi_j^{\mu}$ so $(\Sigma_{\mu}X_{\mu}^{\rho}\Phi_j^{\mu})$ form a basis of $\operatorname{Ker}\lambda$.

Now returning to the more general case of $r \geq 3$, the initial relation in ${}_n\mathcal{O}^{2(n-r)}$ was reduced to $(2k)^{n-r}$ relations in ${}_r\mathcal{O}^{2(n-r)(k+1)^{n-r}}$, whose entries are just the coefficients of δ^N, $z_{r+2}\delta - Q_{r+2}, \cdots, z_n\delta - Q_n$, $P_{r+1}, \cdots, P_n$. These coefficients have been rigged to be either identically zero or regular in each of the directions $z_1, \cdots, z_r$, so we can continue the process of reducing to relations of less variables without making any change of coordinates. There is a new difficulty: whereas previously we had a bound on the length of the resulting sections because $k = K$, we now have no such bound. However we are not really interested in the length of the entire section, but only in the number of terms c_{ν}^{1} that will eventually be used to generate π_1's such that $\delta^N\pi_1 + (z_{r+2}\delta - Q_{r+2})\pi_2 + \cdots + (z_n\delta - Q_n)\pi_{n-r} + P_{r+1}\pi_{n-r+1} + \cdots + P_n\pi_{2(n-r)} = 0$, and it is possible to put a bound on these. Each resulting relation contains either some tail coefficient or 1—hence some entry arising from the decomposition of P's, whose degree in the next variable $\leq \mu(V, p)$. These special entries can be used as the Weierstrass element in the next stage, so by Remark 1, the number of terms used to generate π_1 grows only by a multiplicative factor of $k + 1$ each time. We continue until reaching ${}_0\mathcal{O} = \mathbf{C}$ and read off the dimension of the appropriate vector space.

Remark. If V is an algebraic variety, the generators resulting from this proof are polynomials.

The following example of one-dimensional analytic variety in $\mathbf{C}^3$ whose global ideal is not finitely generated was communicated to me by Bernard Kripke. There is, of course, no such example in $\mathbf{C}^1$ or $\mathbf{C}^2$ since any subvariety of $\mathbf{C}^1$ or $\mathbf{C}^2$ is generated locally by one or two elements.

In $\mathbf{C}^3$, there is, for every n, a variety V such that $I(V)$ tequires at least n generators. To construct such a variety, note that there are $(n+1)(n+2)/2$ monomials in the three variables x, y, z of degree n, e.g. for $n = 2$, they are x^2, xy, xz, y^2, yz, z^2. Thus the vector space H_n of homogeneous polynomials of degree n on $\mathbf{C}^3$ has dimension $k = (n+1)(n+2)/2$ and is spanned by homogeneous polynomials $p_1, \cdots, p_k$. There is no dependence relation of the form $c_1 p_1 + \cdots + c_k p_k \equiv 0$, or in other words, the map: $\mathbf{C}^3 \to \mathbf{C}^k$ defined by $(x, y, z) \to (p_1(x, y, z), \cdots, p_k(x, y, z))$ maps $\mathbf{C}^3$ into no complex hyperplane in $\mathbf{C}^k$. Let $u = (x, y, z)$; we can find $u_1, \cdots, u_k$ such that the vectors $(p_1(u_1), \cdots, p_k(u_1))$, $\cdots$, $(p_1(u_k), \cdots, p_k(u_k))$ are linearly independent in $\mathbf{C}^k$. The rows and columns of the $k \times k$ matrix $[p_i(u_j)]$ are independent so if $f = c_1 p_1 + \cdots + c_k p_k$, then $(f(u_1), \cdots, f(u_k)) = 0 \Leftrightarrow (c_1, \cdots, c_k) = 0$.

In other words, every homogeneous polynomial of degree n which vanishes at $u_1, \cdots, u_k$ vanishes identically. A fortiori, every homogeneous polynomial of degree $< n$ which vanishes at $u_1, \cdots, u_k$ must vanish identically. Now let V be the variety consisting of the k complex lines joining the origin to the points $u_1, \cdots, u_k$ in $\mathbf{C}^3$. If f is any holomorphic function in $I(V)$, then f can be expanded in homogeneous polynomials $f(x, y, z) = \sum_{j=0}^{\infty} h_j(x, y, z)$, where h_j is a homogeneous polynomial of degree j. Since $f(tx, ty, tz) = \sum_{j=0}^{\infty} t^j h_j(x, y, z)$, it follows that each of the h_j vanish on V as well. Therefore $f = \sum_{j=n+1}^{\infty} h_j$. Indeed, every holomorphic function which vanishes on the intersection of V with a neighborhood of the origin is a sum of homogeneous polynomials of degree $> n$.

Now there are $k + n + 2$ homogeneous monomials of degree $n + 1$ on $\mathbf{C}^3$, say $q_1, \cdots, q_{k+n+2}$. If $h \in H_{n+1}$, then $h \in I(V) \Leftrightarrow h(u_1) = \cdots = h(u_k) = 0$. Consider the system of linear equations:

$$c_1 q_1(u_1) + \cdots + c_{k+n+2} q_{k+n+2}(u_1) = 0,$$

$$c_1 q_1(u_k) + \cdots + c_{k+n+2} q_{k+n+2}(u_k) = 0.$$

It defines a subspace of $\mathbf{C}^{k+n+2}$ of dimension $\geq k + n + 2 - k = n + 2$ corresponding to a subspace S of H_{n+1} consisting of homogeneous polynomials $h = c_1 q_1 + \cdots + c_{k+n+2} q_{k+n+2}$ which vanish on V. There are at least $n + 2$ linearly independent homogeneous polynomials of degree $n + 1$ which vanish on V; since every element of $I(V)$ is a sum of homogeneous polynomials of degree $\geq n + 1$, it follows that any set of generators for $I(V)$ must contain at least $n+2$ elements. Suppose $g_1, \cdots, g_m$ generate $I(V)$ and write $g_i = h_i + r_i$, where $h_i \in H_{n+1}$ and r_i is of order $> n+1$. If $h \in H_{n+1}$ and $h = f_1 g_1 + \cdots + f_m g_m$ where $f_i \in {}_n\mathcal{O}_0$, then $h = f_1(0)h_1 + \cdots + f_m(0)h_m$. That is, the vector space $S = H_{n+1} \cap I(V)$ is spanned by $h_1, \cdots, h_m$ so $m \geq n+2$.

Now this proves that there is no bound on the number of generators that may be required for the ideal of a variety in $\mathbf{C}^3$. Let $V_1, V_2, V_3, \cdots$ be a sequence of varieties in $\mathbf{C}^3$ constructed in this way so that $I(V_j)$ requires at least $j+2$ generators and each V_j consists of $(j+2)(j+1)/2$ lines through the origin. Now choose inductively a sequence of vectors $c_1, c_2, c_3, \cdots$ in $\mathbf{C}^3$ as follows: let $c_1 = 0$ and suppose $c_1, \cdots, c_n$ have already been chosen. Each V_j for $j = 1, \cdots, n+1$ consists of a union of complex lines through the origin spanned by a finite set S_j of vectors; let $A_n = S_1 \cup \cdots \cup S_{n+1}$. Each pair of vectors in A_n span a complex 2-dim subspace of $\mathbf{C}^3$; let B_n be the union of all such subspaces and let $E_n = (B_n + c_1) \cup (B_n + c_2) \cup \cdots \cup (B_n + c_n)$ be the union of the translates of B_n through the vectors $c_1, \cdots, c_n$. Now choose c_{n+1} so that $\operatorname{dist}(c_{n+1}, E_n) \geq 1$.

Let $W_j = V_j + c_j$ for $j = 1, \cdots, n+1$ and it follows that $\operatorname{dist}(W_{n+1}, W_1 \cup \cdots \cup W_n) \geq 1$. Indeed, there are $u_{n+1} \in W_{n+1}$ and $u_j \in W_j$ for some $1 \leq j \leq n$ such that $\operatorname{dist}(W_{n+1}, W_1 \cup \cdots \cup W_n) = \|u_{n+1} - u_j\|$. Say $u_{n+1} = c_{n+1} + ta$ for some $a \in S_{n+1}$ and $u_j = c_j + sb$ for some $b \in S_j$. Then $\|u_{n+1} - u_j\| = \|c_{n+1} - (c_j - ta + sb)\|$, but $c_j - ta + sb \in c_j + B_n \subset E_n$. Thus $\|u_{n+1} - u_j\| \geq \operatorname{dist}(c_{n+1}, E_n) \geq 1$.

Finally let $W = \bigcup_{j=1}^{\infty} W_j$; by construction this union is locally finite so W is a variety in $\mathbf{C}^3$. Clearly no finite subset of $\mathcal{O}(\mathbf{C}^3)$ can generate $I(W)$ since at least n functions are required just to generate the germ of $I(W)$ at c_n.

Since W can be embedded in $\mathbf{C}^n$ for every $n > 3$, it is also true that for every $n \geq 3$, there is a variety $W \subset \mathbf{C}^n$ such that $I(W)$ is not finitely generated.

Note that $\mu(W, c_n) = m(W, c_n) = (n+1)(n+2)/2$ and that the minimal and maximal multiplicity are one at all other points of W.

The following construction of an irreducible one-dimensional variety in $\mathbf{C}^3$ whose global ideal is not finitely generated was suggested by James King:

For any integer n, there exists an irreducible space curve X_n [17] whose ideal cannot be generated by fewer than n elements. Then these X_n can be patched together away from the singular points to form a noncompact, irreducible, one-dimensional complex space X. By [7, Theorem IX, B.10], X is a Stein space so [7, Theorem VII, C.10] there is a holomorphic homeomorphism of X into $\mathbf{C}^3$. The obstruction to the existence of an imbedding of X in $\mathbf{C}^3$ is that the local holomorphic imbedding dimension of X be bounded [18]; but this is at most 3, so an embedding exists.

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